Mathematics Part II Dissertation

The Ends of Finitely Generated Groups

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1 Introduction

This project is concerned with the general problem of understanding groups, and in particular finitely generated infinite groups. It is natural to study finitely generated and finitely presented groups; many examples appear throughout mathematics.

Given a finitely generated group G with generating set S, we place a natural metric on G (the *word metric*; see 2.3). This allows us to consider G as a metric space. Unfortunately, the metric depends on the choice of generating set S, rather than just on the group G. However, there are many properties of this metric which do not depend on the choice of S. One such concept is that of the number of *ends* of G. This project is concerned with understanding the ends of a group and studying groups using this concept.

The definition of ends was achieved by Freudenthal in 1931 [4, pp. 692-713]. The application to group theory was initiated by Freudenthal [5, pp. 261-79],[6, pp. 1-38] and Hopf [9, pp. 81-100] and Specker [20, pp. 167-174].

There are many (equivalent) definitions of the *ends* of a group, and our first task is to present these definitions and prove that they are equivalent.

It is immediate from the definition that the number of ends of a group is either a non-negative integer or infinite. We prove that a finitely generated group has either 0,1,2 or infinitely many ends (See Theorem 4.15). The groups with 0 ends are precisely the class of finite groups. We say little about finite groups, as the techniques we develop are largely applicable to infinite groups.

By constructing structure theorems about groups that have two or infinitely many ends the theory can be broken up in to smaller parts. A key theorem is that of Stallings [22]. Note we do not prove Stalling's theorem here, but we do discuss a few of its implications.

Stallings' theorem tells us that any finitely generated group G with infinitely many ends is a free product with amalgamation, where the amalgamated subgroup is finite. In particular if G is a finitely generated torsion free group then G is a free product. Although this second result is included in [22] it was proved earlier in [21].

The motivation for this project is primarily group theoretic but we will need to use some algebraic topology to prove the results. In particular we will need the theory of groups acting on spaces and some elementary cohomology theory. We will use this to analyse some geometric properties of groups.

We start, in Section 2 by defining a metric on a group G and the Cayley graph of G. These ideas will be central to the paper. In Section 3 we then introduce the idea of a quasi-isometry between metric spaces so that these ideas are well defined. A useful result is that the ends of a group is a quasi-isometric invariant.

Once this has been completed we define the ends of a group in Section 5 and prove some important foundations for the structure theorems we want to show. Finally, in Section 6 we prove that the only possibilities for the number of ends of a group are: 0,1,2 or infinity. Using this we can then prove the structure theorems for groups with two ends.

2 Groups as Metric Spaces

We will begin with the most important and basic definitions. Throughout the project we work only with finitely generated groups.

The first definition is that of a simplical (or cell) complex. Simplicial complexes are very useful spaces to work with for this theory.

We use Lackenby's approach [12]. The first thing to consider is the idea of attaching cells to spaces.

Definition 2.1. A space Y is obtained from a space X by *attaching an m-cell* if there exists $f : S^{m-1} \to X$ and Y is the quotient of the disjoint union $D^m \cup X$; where the quotient identifies x and all points in $f^{-1}(\{x\})$ for all $x \in S^{m-1}$.

For example given two copies of S^1 we can form $S^1 \wedge S^1$, the space formed by 'attaching' the circles at a point (Figure 1).

Definition 2.2.

1. Start with a discrete set, X^0, whose points are regarded as 0-cells.
2. Inductively we form the *n-skeleton* X^n from X^{n-1} by attaching $n-$*cells* to X^{n-1}.
3. This process can either continue indefinitely, so that $X = \bigcup\limits_n X^n$, or it can stop giving $X = X^n$.

   An X created in this way is called a *simplicial complex* (or *cell complex*).

A simple example is that of the torus. Consider the usual representation as square with opposite edges identified with the same orientation (Figure 2).

The torus has one vertex, three edges and two faces. That is, we start with one 0-cell (the vertex), attach three 1-cells (the edges) and then attach two 2-cells (faces) along the edges. It's not necessary to attach two 2-cells, but in this way we have a *triangulated* space (see [11]).

I put a picture of the wedge of 2 circles here, but it isn't working for pdf.

Figure 1: The Wedge of two Circles

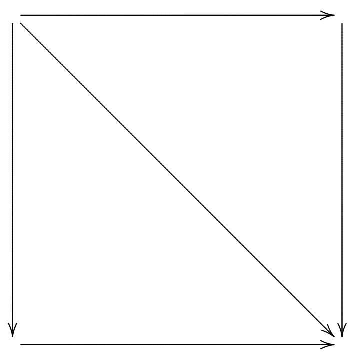

Figure 2: The (triangulated) Torus

Similarly, we can triangulate $\mathbb{R}^2$ (Figure 3)to obtain the following simplicial complex:

Definition 2.3. Given a group G and a finite generating set S we can define the *word metric on G*. Given $g, h \in$G there exists a finite sequence: $(s_1, s_2, s_3, \dots, s_{k-1}, s_k)$with the $s_i \in S \cup S^{-1}$ and $g^{-1}h = s_1 s_2 \dots s_k$ However, this is not unique. Choosing a some sequence of minimal length we can define $d_S(g, h) = k$.

Lemma 2.4. *The pair* (G, d_S) *is a metric space.*

Proof:

(i)Positivity: Clearly $d_S(g, h) \geq 0$ and $d_S(g, h) = 0$ if and only if $g = h$.

(ii)Symmetry: By replacing elements of S with the corresponding elements of S^{-1} and vice versa, then reversing the order of the s_i we see that $d_S(g, h) = d_S(h, g)$.

(iii)Transitivity: Now, if $\prod s_i = g^{-1}h$ and $\prod s_i' = h^{-1}k$ then $(\prod s_i)(\prod s_i') = g^{-1}k$. So, by concatenating products we see that $d_S(g, h) + d_S(h, k) \geq d_S(g, k)$. □

Cayley Graphs

Having defined a metric on G we can ask many interesting topological questions about groups. We start with some basic definitions.

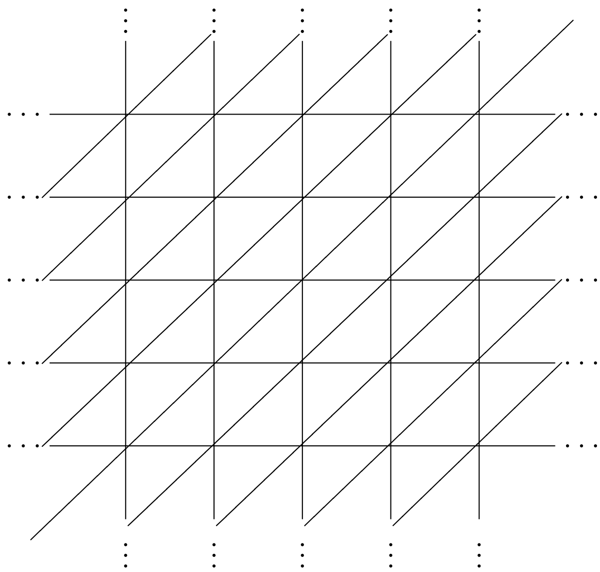

Figure 3: $\mathbb{R}^2$ as a simplicial complex

Definition 2.5. Given a group Γ and a generating set S we can define the *Cayley Graph*, Γ_S of Γ. The Cayley graph is a graph with orientation with vertex set Γ. The edge set of Γ_S is E, where $e \in E$ if and only if there exist $g, h \in \Gamma$ and $s \in S \cup S^{-1}$ with $s = g^{-1}h$, and e joins g to h.

We define a natural metric on Γ_S as follows; let each edge be isomorphic to I the unit interval in $\mathbb{R}$, and give Γ_S the *path metric*, namely, the distance between two points is the length of the shortest path between them.

Note that the distance between two points joined by a generator is 1. So, given two vertices of Γ_S the minimum distance between them, the length of the shortest path, is then exactly the minimum number of generators needed for gh^{-1}. That is, restricting the metric on Γ_S to the vertices gives d_S, the word metric on G with respect to S.

A useful example is $\mathbb{Z} \times \mathbb{Z}$, which can be drawn as in Figure 4 to highlight that it is the Cayley graph of a *group*, $\mathbb{Z} \times \mathbb{Z}$. Or, more usually, to highlight its geometric structure in Figure 5.

Also, consider $F\langle a, b\rangle$ (Figure 6) the free group on 2 generators. This group is interesting, both from the point of being a group with infinitely many ends, and also in algebraic topology.

Unfortunately, the Cayley graph of Γ is not unique for each group Γ but is defined for each pair (Γ, S) where S is a finite generating set for Γ. See,

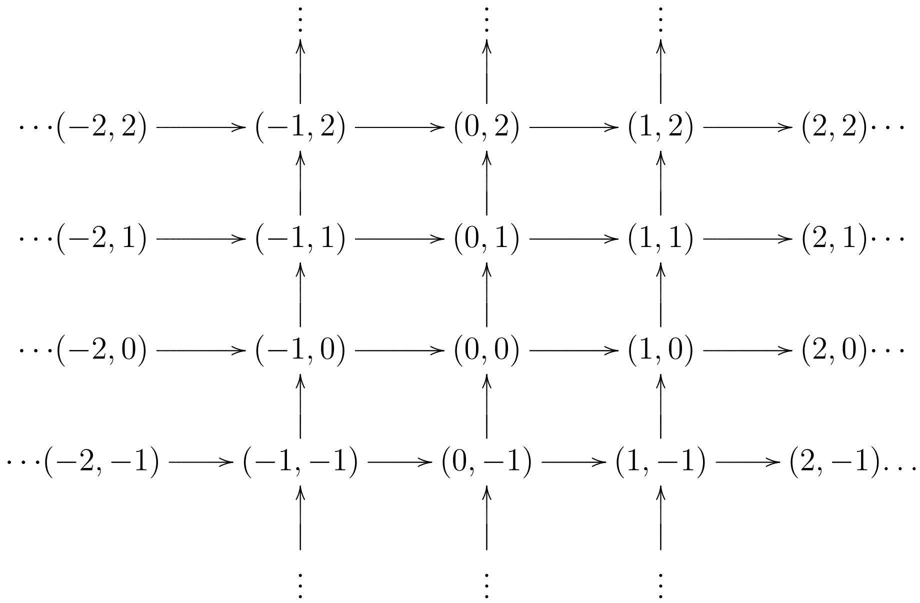

Figure 4: The Cayley graph of $(\mathbb{Z} \times \mathbb{Z}, \{(0,1),(1,0)\})$ (a)

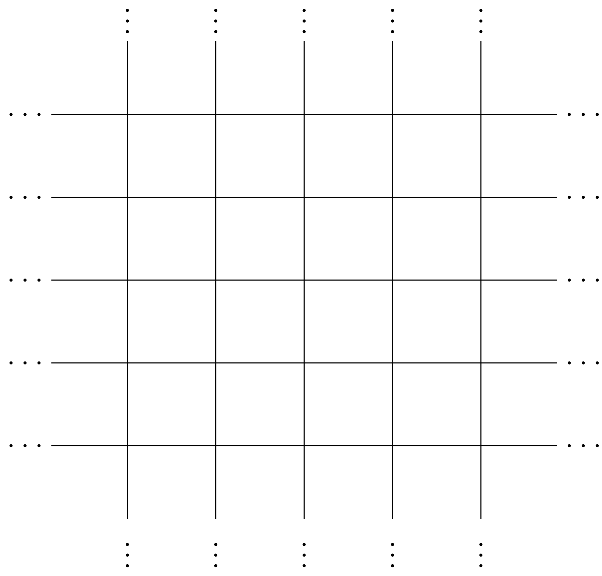

Figure 5: The Cayley graph of $(\mathbb{Z} \times \mathbb{Z}, \{(0,1),(1,0)\})$ (b)

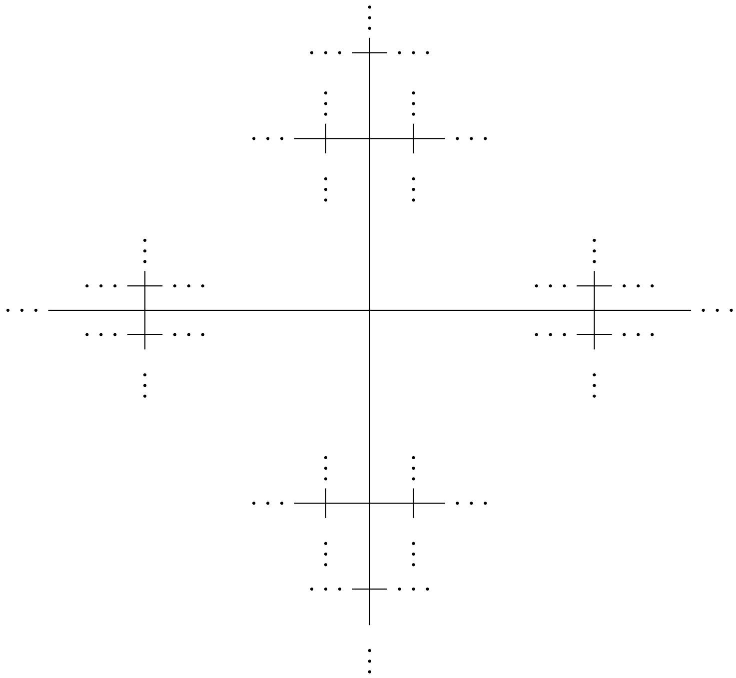

Figure 6: The Cayley graph of $(F\langle a,b\rangle, \{a,b\})$

$$\cdots \longrightarrow -4 \longrightarrow -3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

Figure 7: The Cayley graph of $(Z, \{1\})$

for example, the Cayley graphs of $(\mathbb{Z}, \{1\})$ and $(\mathbb{Z}, \{2,3\})$ (Figures 7 and 8). Also, we see that, for example $d_{\{1\}}(1,2) = 1$ but $d_{\{2,3\}}(1,2) = 3$.

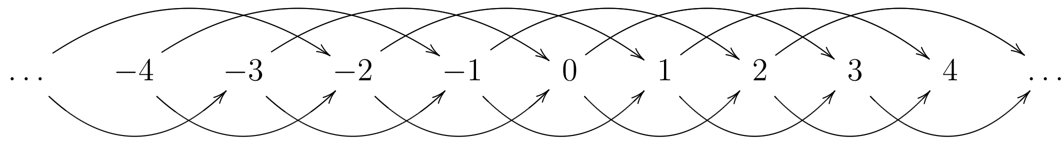

Figure 8: The Cayley graph of $(\mathbb{Z}, \{2,3\})$

However, in Lemma 3.18 we show that this ambiguity is not important in this paper.

3 Quasi-isometries

Definition 3.1. If X_1, X_2, are metric spaces, a (not necessarily continuous) map, $f : X_1 \to X_2$ is said to be a *quasi-isometry* if

1. There exists $K \geq 1, C \geq 0$ so that for all x, y in X_1

$$\frac{1}{K} d_1(x,y) - C \leq d_2(f(x), f(y)) \leq K d_1(x,y) + C.$$

2. There is a constant, $M \geq 0$, such that every point of X_2 lies within an M neighbourhood of the image of f. In this case X_1 and X_2 are said to be *quasi-isometric*.

Lemma 3.2. *Being quasi-isometric is an equivalence relation.*

Proof: 1. Reflexivity is obvious as the identity map is sufficient.

2. Symmetry: Suppose f is a quasi-isometry from (X_2, d_1) to (X_2, d_2). Now, we know that every point of X_2 lies within an M neighbourhood of the image of f. Define $g : X_2 \to X_1$ by:

   (i) If $x \in \text{Im}(f)$ then define $g(x)$ to be some element of $f^{-1}(x)$.

   (ii)If $x \notin \text{Im}(f)$ then set $g(x) = q$, where $f(q)$ is some point and $q = g(f(q))$, with $d_2(x,q) \leq M$ of x. We can do this as f is a quasi-isometry.

   Suppose $x, y \in X_2$, then there exist points x_f, y_f in the image of f with $g(x) = g(x_f)$ and $g(y) = g(y_f)$. By definition of g these satisfy $d_2(x, x_f) \leq M$ and $d_2(y, y_f) \leq M$.

   By the triangle inequality $d_2(x,y) \leq d_2(x_f, y_f) + 2M$ and $d_2(x,y) \geq d_2(x_f, y_f) - 2M$.

For all a, b in X_1 we have,

$$\frac{1}{K}d_1(a,b) - C \leq d_2(f(a), f(b)) \leq Kd_1(a,b) + C,$$

thus we see that

$$\frac{1}{K}d_1(g(x_f), g(y_f)) - C \leq d_2(x_f, y_f) \leq Kd_1(g(x_f), g(y_f)) + C.$$

So,

$$d_2(x,y) \leq d_2(x_f, y_f) + 2M \leq Kd_1(g(x_f), g(y_f)) + C + 2M,$$

and

$$d_2(x,y) \geq d_2(x_f, y_f) - 2M \geq \frac{1}{K}d_1(g(x_f), g(y_f)) - C - 2M.$$

So, g is a quasi-isometry from X_2 to X_1 and that being 'quasi-isometric' is a symmetric relation.

3. Transitivity: Suppose (X_1, d_1) is quasi isometric to (X_2, d_2) and (X_2, d_2) quasi-isometric to (X_3, d_3). Then there exist quasi-isometries

$$f : X_1 \to X_2, \;\; g : X_2 \to X_3.$$

Taking $h = g \circ f$ we immediately see that h is a quasi-isometry from X_1 to Z_1 with:

$$\Big(\frac{1}{K_1K_2}\Big)d_1(a,b) - \Big(\frac{C_1}{K_2} + C_2\Big) \leq d_3(h(a), h(b)) \leq (K_1K_2)d_1(a,b) + (K_2C_1 + C_2),$$

for all a, b in X_1.

Also every point of X_3 lies within a $(K_2M_1 + C_2) + M_2$ neighbourhood of the image of h. Here K_1, C_1, M_1 are the quasi-isometry constants and K_2, C_2, M_2 are the quasi-isometry constants for g. □

Proposition 3.3. *If G is a group and S, S' are two finite generating sets, then the identity map, $i : (G, d_S) \to (G, d_{S'})$, is a quasi-isometry.*

Proof: As S is a generating set we can write the elements of S' in terms of elements of $S \cup S^{-1}$ so that

$$\begin{aligned} S' &= \{s'_k : 0 \leq k \leq l\} \\ &= \{s_{i_1}s_{i_2}\dots s_{i_{m(k)}} : s_{i_1}s_{i_2}\dots s_{i_{m(k)}} = s'_k, 0 \leq k \leq l, s_{i_j} \in S \cup S^{-1}, m(k) \in \mathbb{N}\}. \end{aligned}$$

Suppose $g, h \in G$. Let $K_1 = \max_{0 \leq k \leq l}\{m(k)\}$ in the above. We see that $d_{S'}(g,h) \leq K_1 d_S(g,h)$. Similarly, writing the elements of S in terms of S' we get K_2 with $d_S(g,h) \leq K_2 d_{S'}(g,h)$. Let $K = \max\{K_1, K_2\}$.

Now,

$$\frac{1}{K} d_S(g,h) - 0 \leq d_{S'}(i(g), i(h)) \leq K d_S(g,h) + 0,$$

which is exactly what we need for i to be a quasi-isometry. (Note that, as the identity is surjective, $M = 0$.) So, for this paper, it now makes sense to talk about *the* word metric. □

Natural Quasi-isometries and the Švarc-Milnor Lemma

We follow Bridson and Haefliger [1, pp 4, 131-145] for this section.

We prove that a group acting on space in a particular way gives rise to a quasi-isometry between the group and the space it acts on. Later we will note that the number of ends of a group is a quasi-isometry invariant; this tells us that if a group acts on a space in this way it has the same number of ends as that space.

Definition 3.4. Let (X, d) be a metric space and x, y two points in X. A *geodesic path* joining x to y (or, a geodesic from x to y) is a map c from a closed interval, $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. So, in particular, $l = d(x, y)$). If $c(0) = x$ then c is said to *issue* from x.

In the above definition the image α of c is called a *geodesic segment* with endpoints x and y.

Definition 3.5. A metric space (X, d) is said to be a *geodesic metric space* (or a *geodesic space*) if every two points in X are joined by a geodesic path.

Remark 3.6.

1. The Cayley graph, Γ_S of a group,G, is a geodesic metric space.
2. When G acts on Γ_S, its Cayley graph, it acts properly by isometries.
3. If G is finitely generated then Γ_S is a proper geodesic metric space.

Definition 3.7. An *action* of a group Γ on a topological space X is a homomorphism $\phi : \Gamma \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ is the group of self homeomorphisms of X. If X is a metric space, then one says that Γ is *acting by isometries* on X if $\phi(\Gamma) \subset \text{Isom}(X)$, where $\text{Isom}(X)$ is the group of all isometries from X to itself.

We shall now suppress all mention of ϕ and write $\gamma.x$ for the image of $x \in X$ under $\phi(\gamma)$, and $\gamma.Y$ for the image of a subset $Y \subset X$. we shall write $\Gamma.Y$ to denote $\bigcup_{\gamma \in \Gamma} \gamma.Y$.

Definition 3.8. An action is said to be *free* if, for every $x \in X$ and every $\gamma \in \Gamma \setminus \{1\}$, one has $\gamma.x \neq x$.

Remark 3.9. Note that the natural action of G on Γ_S is free and, therefore, so is the action of H, for any subgroup,H, of G.

Definition 3.10. An action is said to be *cocompact* if there exists a compact set $K \subset X$ such that $X = \Gamma.K$.

Definition 3.11. Let Γ be a group acting by isometries on a metric space X. The action is said to be *proper* (or, Γ acts properly on X) if for each $x \in X$ there exists a number $r > 0$ such that the set,

$$P = \{\gamma \in \Gamma : \gamma.B(x, r) \cap B(x, r) \neq \emptyset\}$$

is finite.

Remark 3.12. If X is proper then, for any $r > 0$ the set P is finite.

Definition 3.13. Given an action of a group Γ on a space X, we write Γ_x for the stabiliser (isotropy subgroup) of $x \in X$, that is $\Gamma_x = \{\gamma \in \Gamma : \gamma.x = x\}$.

Lemma 3.14. *Let X be a topological space, let Γ be a group acting on X by homeomorphisms, and let $U \subset X$ be an open subset such that $X = \Gamma.U$.*

If X is connected , then the set $S = \gamma \in \Gamma : \gamma.U \cap U \neq \emptyset$ generates Γ.

Proof: Let $H \subset \Gamma$ be the subgroup of Γ generated by S, let $V = H.U$ and let $V' = (\Gamma \setminus H).U$. If $V \cap V' \neq \emptyset$, then there exist $h \in H, h' \in \Gamma \setminus H$ such that $h^{-1}h'.U \cap U \neq \emptyset$ and hence $h' \in HS \subset H$, contrary to assumption. Thus the open sets V and V' are disjoint. Now, V is non-empty and $X = V \cup V'$, so since X is connected, we have $V' = \emptyset$. Therefore $H = \Gamma$, as required. □

Lemma 3.15. *Let (X, d) be a metric space. Let Γ be a group with finite generating set A and associated word metric d_A. If Γ acts by isometries on X, then for every choice of base point $x_0 \in X$ there exists a constant $\mu > 0$ such that $d(\gamma.x_0, \gamma'.x_0) \leq \mu d_A(\gamma, \gamma')$ for all $\gamma, \gamma' \in \Gamma$.*

Proof: Let $\mu = \max\{d(x_0, a.x_0) : a \in A \cup A^{-1}\}$. If $d_A(\gamma, \gamma') = n$ then $\gamma^{-1}\gamma' = a_1 a_2 \ldots a_n$ for some $a_j \in A \cup A^{-1}$. Let $g_i = a_1 a_2 \ldots a_i$. By the triangle inequality, $d(\gamma.x_0, \gamma' x_0) =$

$$d(x_0, \gamma^{-1}\gamma'.x_0 \leq d(x_0, g_1.x_0) + d(g_1.x_0, g_2.x_0) + \ldots + d(g_{n-1}.x_0, \gamma^{-1}\gamma'.x_0).$$

Now, for each i we have $d(g_{i-1}.x_0, g_i.x_0) = d(x_0, g_{i-1}^{-1} g_i.x_0) = d(x_0, a_i.x_0) \leq \mu$.
□

The Švarc-Milnor Lemma provides many natural examples of quasi-isometries. This is especially useful as we show that the number of ends of a space is a quasi-isometric invariant.

Theorem 3.16. (The Švarc-Milnor Lemma). *Let X be a proper geodesic space. If Γ acts properly and cocompactly by isometries on X, then Γ is finitely generated and for any choice of basepoint $x_0 \in X$, the map $\gamma \to \gamma.x_0$ is a quasi-isometry for any word metric on Γ.*

Proof: Let $C \subset X$ be a compact set with $\Gamma.C = X$. Choose $x_0 \in X$ and find $D > 0$ such that $C \subset B(x_0, D/3)$ and let $A = \{\gamma \in \Gamma : \gamma.B(x_0, D) \cap B(x_0, D) \neq \emptyset\}$. As Γ acts properly and X is proper, A is finite.

In Lemma 3.14 we showed that A generates Γ. Let d_A be the word metric on (note, which metric we choose is irrelevant) Γ associated to A. Lemma 3.15 yields a constant μ such that $(\gamma.x_0, \gamma'.x_0) \leq \mu d_A(\gamma, \gamma')$ for all $\gamma, \gamma' \in \Gamma$, so it only remains to bound $d_A(\gamma, \gamma')$ in terms of $(\gamma.x_0, \gamma'.x_0)$.

Note that both metrics are Γ-invariant. So we need only compare $d_A(1, \gamma)$ and $d(x_0, \gamma.x_0)$.

Given $\gamma \in \Gamma$ and a geodesic path $c : [0, 1] \to X$ of finite length with $c(0) = x_0$ and $c(1) = \gamma x_0$, we can choose a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$ such that $d(c(t_i), c(t_{i+1})) \leq D/3$ for all i.

For each t_i we choose an element $\gamma_i \in \Gamma$ such that $d(c(t_i), \gamma_i.x_0) \leq D/3$; choose $\gamma_0 = 1$ and $\gamma_n = \gamma$. Then, for $i = 1, \ldots, n$ we have $d(\gamma_i.x_0, \gamma_{i-1}.x_0) \leq D$ and hence $a_i := \gamma_{i-1}^{-1}\gamma_i \in A$.

$$\gamma = \gamma_0(\gamma_0^{-1}\gamma_1) \ldots (\gamma_{n-2}^{-1}\gamma_{n-1})(\gamma_{n-1}^{-1}\gamma_n) = a_1 \ldots a_{n-1} a_n.$$

Because X is a geodesic space, we can choose the curve c considered above to have length less than $d(x_0, \gamma.x_0) + 1$. If we take as coarse a partition, $0 = t_0 < t_1 < \ldots < t_n = 1$ as possible with $d(c(t_i), c(t_{i+1})) \leq D/3$, then $n \leq \frac{3}{D} d(x_0, \gamma.x_0) + \frac{3}{D} + 1$. Since γ can be expressed as a word of length n, we get $d_A(1, \gamma) \leq \frac{3}{D} d(x_0, \gamma.x_0) + \frac{3}{D} + 1$. □

Consider the natural action of $(\mathbb{Z} \times \mathbb{Z}, d_S)$ on $\mathbb{R}^2$ where $\mathbb{R}^2$ is endowed with the usual Euclidean metric. That is,

$$(m, n) : (x, y) \to (x + m, y + n).$$

This action is clearly an isometry. Also, if we consider the closed subspace, $I \times I$ (where I is the closed unit interval in $\mathbb{R}$), we see that $(\mathbb{Z}\times\mathbb{Z}).(I \times I) = \mathbb{R}^2$ so the action is cocompact. If we consider an open ball of radius $\frac{1}{3}$ around any point we see that the action is proper.

So, using Švarc-Milnor we can easily see that $(\mathbb{Z}\times\mathbb{Z}, d_S)$ is quasi-isometric to $\mathbb{R}^2$.

Considering the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ (See Figure 5) then helps with a intuitive understanding of Švarc-Milnor.

There is a a more general result concerned with $\pi_1(X)$, the fundamental group, of a space. If X be a compact metric simplicial complex then the universal covering space $\widetilde{X}$ has an induced metric. Now, $\pi_1(X)$ acts on $\widetilde{X}$ and this action satisfies the criteria of the Švarc-Milnor Lemma. Hence $\pi_1(X)$, with any word metric, is quasi-isometric to $\widetilde{X}$.

Remark 3.17. Inclusion is a quasi-isometry from (G, d_S) to Γ_S. The quasi-isometry constants are $K = 1, C = 0, M = \frac{1}{2}$.

Lemma 3.18. *Let G be a group and let S, S' be finite generating sets for G, then the Cayley graphs Γ_S and $\Gamma_{S'}$ are quasi-isometric.*

Proof: Using Remark 3.17 and the fact that quasi-isometry is a symmetric relation (Lemma 3.2) we have that Γ_S is quasi-isometric to (G, d_S). Also, (G, d_S) is quasi-isometric to $(G, d_{S'})$ by Proposition 3.3. Now, using Remark 3.17 again we have $(G, d_{S'})$ is quasi-isometric to $\Gamma_{S'}$. Now, quasi-isometry is an equivalence, by relation Lemma 3.2, this shows that Γ_S and $\Gamma_{S'}$ are quasi-isometric. □

We have shown that any 2 Cayley graphs are quasi-isometric (for finite generating set S). So, for the purposes of quasi-isometrically invariant properties of groups we may now talk about *the* Cayley graph of G.

Lemma 3.19. *Let G be a finitely generated group and H a subgroup of finite index in G. Then (G, d_S) is quasi-isometric to $(H, d_{(S\cap H)})$.*

Proof: Let H act on (G, D_S). We have noted in Remark 3.9 that H acts freely and hence properly. The subgroup H has finite index in G so H acts with finite quotient, and hence cocompactly. So, Theorem 3.16 implies that (G, d_S) is quasi-isometric to $(H, d_{(S\cap H)})$. □

4 Homology

Central to many of the arguments in the paper are the ideas of cohomology. Only simplicial cohomology is needed in the paper which simplifies the theory.

We assume a familiarity with homology (See, for example [12], [13, pp. 147-178] or [10, Chapter 2]) but remind the reader of a few important ideas.

The $n^{\text{th}}-$homology groups are often described as giving the n dimensional information about a space. Although not precise a common idea is that if a space has only n dimensional parts has trivial $m^{\text{th}}-$homology groups for all $m > n$. Similarly, for cohomology, a space with only $n-$dimensional parts has trivial $(m+1)^{\text{st}}-$cohomology groups for all $m > n$.

In this project the spaces we work with are all simplicial complexes. In this case the above can be made precise. If , in the construction of X, as in the definition of a simplicial complex, only cells of dimension less than than n are added to X then X has trivial $m^{\text{th}}-$cohomology groups for all $m > n$.

Hence for Cayley graphs, with only 0 and 1 dimensional cells, cohomolgy is greatly simplified. We see that, for this paper, we need only consider the $0^{\text{th}}-$homology group.

We now define homology and cohomology for simplicial complexes.

The first step in considering homology is to create a chain complex. The space is split in to a countably infinite list of abelian (chain) groups. Each of the chain groups contains $n-$*chains*. These $n-$chains are linear combinations of simplices and the preferred simplices here are generalisations of triangles.

To make use of this chain complex we introduce maps from the nth chain group to the $(n-1)^{st}$ chain group.

Definition 4.1. We define $\partial_n : C_n(X) \to C_{n-1}(X)$, the *boundary* map, by:

$$\partial_n(c) := \sum_i (-1)^i f_i(c).$$

where the f_i are the i^{th} face maps. These are defined on the algebraic topology course [12] (or see [13, pp. 159] or [10, p. 7]), so we do not define them here. Intuitively these give the components which make up the edges of a simplex. So, for a tetrahedron the i^{th} face map gives one of the conventional faces. For an edge the faces are the points at both ends. For a triangle a face would be one of its three bounding edges. Recall that in homology theory each simplex is given an *orientation*.

The boundary map then gives us an ordered sum of all of these faces. So, given a triangle, the boundary of the triangle is an ordered sum of the three bounding edges. The boundary of a tetrahedron is an ordered sum of the faces (triangles) of the tetrahedron.

Definition 4.2.

1. Elements of the kernel of a boundary map are the $n-$*cycles*.

2. An element of the image of a boundary map is called a *boundary*.

Remark 4.3. Recall that in any chain complex the boundary maps always satisfy $\partial_{n-1}\partial_n = 0$ for all $n \in \mathbb{N}$. This fact is often written as $\partial\partial = 0$, and this implies that

$$\operatorname{Im}(\partial_{n+1}) \subset \operatorname{Ker}(d_n).$$

As homology groups are abelian this allows us to consider the quotient group $\operatorname{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1})$ and make the following definition.

Definition 4.4. We define the n^{th} homology group of a space,X, as

$$H_n(X) := \operatorname{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1}).$$

Cohomology

Cohomology can be viewed as a dualisation of homology. Instead of working with simplices with cohomology we work with homomorphisms from the simplices to a group G.

The following definitions are from Massey [13, pp.305-307]

Definition 4.5. A *cochain complex* K consists of a sequence of abelian groups $\{K^n\}$ and homomorphisms $\delta^n : K^n \to K^{n+1}$ defined for all n and subject to the condition that $\delta^{n+1}\delta^n = 0$ for all n.

The above definition is very general but we will only use the specific cohomology given by the following dualisation of homology.

Definition 4.6. For any space, X and any abelian group G define $C^n(X, G) := \operatorname{Hom}(C_n(X), G)$.

In this paper the group G will always be $\mathbb{Z}_2$, the additive integers mod 2. So the cochains we work with are homomorphisms from our simplicial complex to $\mathbb{Z}_2$. That is, we assign 0 or 1 to each simplex. Such an assignment describes a subset (of the simplicial complex), namely all the simplices which are mapped to 1.

We also only use a very small amount of cohomology theory. Our spaces will be simplicial complexes, and, for the most part, Cayley graphs.

The maps from the vertices to $\mathbb{Z}_2$ correspond to subsets of G. The chain corresponding to a subset A will map vertices to 1 if that vertex corresponds to an element of A otherwise the vertex is mapped to 0.

Definition 4.7. Define $\delta^q : C^q(X,G) \to C^{q+1}(X,G)$ in following way: given $c \in C_q(X)$ and a homomorphism $f \in \operatorname{Hom}(C_q, G)$, define $\delta^q \in \operatorname{Hom}(C_{q+1}, G)$ by

$$(\delta^q f)(c) := f(\partial_{q+1}(c)).$$

The maps δ^q are called *coboundary operators.*

Now we can define:

$$C^*(X,G) := \{C^n(X,G), \delta^n\},$$

and this is a *cochain* complex.

The above definitions give rise to a structure which has much in common with homology. The condition that $\delta\delta = 0$ allows us to define two useful terms.

Definition 4.8. Elements of the image of a map δ are called *coboundaries*

Definition 4.9. Elements of the kernel of a map δ are called *cocycles*

Definition 4.10. Then the $n^{\text{th}}-$homology group is the quotient of the cocycles by the coboundaries, that is:

$$H^n(X) := \operatorname{Ker}(\delta^n)/\operatorname{Im}(\delta^{n+1}).$$

Note that if we replace n by $-n$ we get a chain complex and can apply the results of homology.

5 Definitions and equivalence of definitions

In the following section we develop the theory of the *ends* of a space. The basic structure of this section mimics a paper by Scott and Wall [18, pp. 171-179]. Also, the proofs used here are based on those given in that paper. We assume throughout that X is a locally finite simplicial complex.

Definition 5.1. Let X be a locally compact simplicial complex. For each compact sub-complex K, the number of connected components of $X \setminus K$ is finite; denote by $n(K)$ the number of components having noncompact closure in X . Now define the number of ends $e(X) := \sup\{n(K)\}$ where the supremum is taken over all compact subcomplexes K.

Note that homeomorphic spaces must have the same number of ends.

Remark 5.2. This can be defined in terms of finite K and finite components of $X \setminus K$ where K finite means that K consists of finitely many simplices.

Clearly $e(X) = 0$ if and only if X is compact. Otherwise $e(X)$ is a positive integer or $+\infty$.

The following spaces become simplicial complexes after triangulation so the definition applies to them.

Remark 5.3. We give examples of complexes with 1, 2, and infinitely many ends.

1. Consider the real line $\mathbb{R}$; removing a single point, k leaves two distinct infinite componenets so $n(\{k\}) = 2$. However any compact K is contained in a closed interval J, and $\mathbb{R} \setminus J$ has only two components, call them C_1 and C_2. Given any closed component, A, of $\mathbb{R} \setminus K$ with no points in either of the C_i we see that $A \subset J$ so A is bounded hence compact.

   This gives $n(\mathbb{R} \setminus K) = 2$ for all compact $K (\neq \emptyset)$ so, $e(\mathbb{R}) = 2$

2. Similarly since the complement of a (large) disc is connected in $\mathbb{R}^n$, $e(\mathbb{R}^n) = 1$.

3. Consider the Cayley Graph of $F\langle a, b\rangle$ (Figure 6). Removing the vertex corresponding to the identity gives 4 infinite components. We can also see that removing all vertices corresponding to elements g with $d(1, g) \leq n$ gives 4×3^n distinct infinite components. Hence $e(F\langle a, b\rangle) = \infty$. It is easy to see that this extends to any free group of rank at least 2.

Consider any locally compact simplical complex X and finite subcomplex K. The (open) subcomplex $\text{st}(K)$, consisting of all simplices with a vertex in K is finite. note that for K, L finite if K is a subset of L then $n(L) \geq n(K)$. So, we have that $n(\overline{\text{st}(K)}) \geq n(\text{st}(K)) \geq n(K)$.

Now, any point of $(X \backslash \text{st}(K)$ can be joined by a path avoiding $\text{st}(K)$ to a vertex in $X \setminus K$. Thus the non-compact connected components are determined by the 1-skeleton of X. So, when computing $e(X)$ we may ignore all cells of dimension > 1, and work in the $1-$skeleton.

The *supports* of a cochain are those simplices which are mapped to a non-zero element of G.

We write $C^*(X)$ for the cochain complex $C^*(X, \mathbb{Z}_2)$. Now, $C^*(X)$ contains a subcomplex $C^*_f(X)$ of cochains with finite support, that is cochains with all but finitely many simplices mapped to zero in $\mathbb{Z}_2$. Note that $C^*_f(X)$ is closed under the coboundary operator as X is locally finite. Write $C^*_e(X)$ for the quotient complex, and $H^*_e(X), H^*_f(X)$ for the cohomology groups of $C^*_e(X), C^*_f(X)$. The short exact sequence

$$0 \to C^*_f(X) \to C^*(X) \to C^*_e(X) \to 0,$$

induces the long exact sequence

$$\ldots \to C_f^n(X) \to C^n(X) \to C_e^n(X) \to C_f^{n+1}(X) \to \ldots,$$

of homology groups. See, for example [12] or [13, pp. 171, 257-259]

As we are only considering the 1-dimensinal structure the quotient cochain complex is

$$0 \xrightarrow{\delta^0} C_e^0 \xrightarrow{\delta^1} C_e^1 \xrightarrow{\delta^2} \ldots \tag{1}$$

So $Im(\delta^0)$ is just the trivial group and $Ker(\delta^1)$ is the set of all finite 0 chains with finite coboundary, hence

$$H_e^0(X) = Ker((\delta^0)/(Im(\delta^{-1})) = \mathrm{Ker}(\delta^0).$$

However, we can write the 0-cochains with finite coboundary as $(\delta^1)^{-1}(C_f^1(X))$. Thus, as we are working in the quotient complex by finite cochains, we have

$$H_e^0(X) = \frac{(\delta^1)^{-1}(C_f^1(X))}{C_f^0(X)} \tag{2}$$

Proposition 5.4. *Let X be a locally finite simplicial complex. Then e(X) is the dimension of $H_e^0(X)$ over $\mathbb{Z}_2$.*

Proof: By above $H_e^0(X) = (\delta^1)^{-1}(C_f^1(X))/C_f^0(X)$, the quotient of 0-cochains with finite coboundary by finite 0-cochains. Now by the argument given above, to calculate the number of ends of X we need only consider the 1 skeleton of X. That is, consider X as consisting of only $1-$cells and $0-$cells.

Let $c_1, \ldots, c_n$ define linearly independent elements of $H_e^0(X)$ (so each of these c_i gives a linearly independent element in the quotient of cochains with finite cochains). As each δc_i is finite, we can choose a finite subgraph K of X containing the supports of all δc_i. That is, we can collect all of the finitely many edges in the δc_i's and create a connected $1-$dimensional subcomplex K containing all of these edges.

For every edge, $e \notin K$ each c_i takes the same value at both ends of e. This is because $\delta(c_i)(e)$ is the directed difference in the values of c_i at the two vertices of e. So, for each connected component A of $X \setminus K$, each c_i takes a constant value $c_i(A)$ on the vertices of A. (As any non zero edge is in K).

If there were only $r < n$ infinite components A_j, then, as each A_j has constant value on all of its vertices we only have r independent blocks of 1's and 0's so if $n > r$ then there is a nontrivial relation $\Sigma\lambda_i c_i(A) = 0$.

This means that $\Sigma\lambda_i c_i$ would be a finite cochain contradicting our choice. (We chose the c_i to be linearly independent in $H_e^0(X)$)

Hence, there must be at least n distinct infinite components after removing K. So we have $n \leq dim H_e^0(X)$ implies $e(X) \geq n$

Conversely, if $e(X) \geq n$ we choose K finite with $n(K) \geq n$, and let $A_1, A_2, \ldots, A_n$ be distinct infinite components of $X - K$. Define the cochain c_i to take the value 1 on vertices of A_i and 0 on other vertices of X. Then if $\delta c_i(e) = 1$, e has one end in A_i, the other in K, by construction of A_i, K. So, e is one of the finitely many edges of $st(K)$. So each δc_i is finite by construction and (as the A_i's differ pairwise on infinitely many vertices) the c_i are independent modulo finite cochains. Hence $H_e^0(X) \geq n$. Therefore $e(X) = \dim_{\mathbb{Z}_2}(H_e^0(X))$, completing the proof of Proposition 5.4 □

We will now begin a different, entirely group theoretic theory of ends.

Definition 5.5. Given two subsets S, T of the same set U, *Boolean addition*, or *symmetric difference* is defined as follows:

$$S + T := \{x : (S \cup T) \setminus (S \cap T)\}.$$

Definition 5.6. Let G be a group.

Let PG be the power set (set of subsets) of G.

Let FG be the subset of PG of finite subsets of G.

Define $QG = \{A \subset G : \forall g \in G, A + Ag \text{ is finite}\}$.

These sets become groups of exponent 2 under Boolean addition.

We refer to sets, A and B, whose difference lies in FG as *almost equal* and write A $\overset{a}{=} B$. This amounts to equality in the quotient group PG/FG.

Moreover, G acts by (right) translation on these groups and QG/FG is the subgroup of elements invariant under this action. Elements of QG are said to be *almost invariant*.

For $A \subset G$, let A^* denote $G \setminus A$, the complement of A in G.

Proposition 5.7. *Let $A \subset G$ be an almost invariant set. Then A^* is also almost invariant.*

Proof: Suppose $x \in A^* + A^*g$, which is to say

$$x \in (G \setminus A) + (G \setminus A)g = (G \setminus A) + (G \setminus (Ag)).$$

That is x is in one of $(G \setminus A)$ and $(G \setminus (Ag))$. This is equivalent to x being in one of A and Ag, ie. $x \in A + Ag$. However this is finite, so A^* is almost invariant. □

We now come to the group theoretic definition of ends:

Definition 5.8. We define the number of ends of G to be

$$e(G) = \dim_{\mathbb{Z}_2}(\frac{QG}{FG}.)$$

Remark 5.9. We can now see:

1. If G is finite then all subsets are finite so we immediately get $e(G) = 0$.
2. Otherwise, G is an infinite set which is invariant (not just almost invariant), so $e(G) \geq 1$.

For finitely generated groups we can identify this new definition of ends with the first. Choose a finite generating set S, and form the Cayley graph Γ_S. Clearly this is locally finite and so the first definition of ends applies.

Proposition 5.10. $e(G) = e(\Gamma_S)$ *for any finite generating set* S.

Proof: Identifying vertices of Γ_S with elements of G, we see a correspondence of $C^0(\Gamma_S)$ with PG and $C^0_f(\Gamma_S)$ with FG. We will show that, if the 0-cochain c corresponds to the subset A, then δc is finite if and only if $A \in QG$.

Now, δc is supported by the set of edges $\{(g, gs) : g \in G, s \in S\}$ with just one end in A. For fixed s, this means that g belongs to just one of A and As^{-1}, that is, $g \in A + As^{-1}$. If A is almost invariant, for each s we have finitely many g. The group G is finitely generated so we have a finite number of edges in total and $A \in QG$ implies that δc is finite.

Conversely if δc is finite, then there are only finitely many edges with one end in A and one end not in A. Each generator s represents one of these edges so As differs from A in only finitely many places. Hence, for $g \in S \cup S^{-1}$, the class of A in PG/FG is invariant under g. Also, c_s corresponding to As has only finite coboundary.

We proceed by induction. Suppose for any g with $d_S(1, g) = p$ we have the class of A is invariant under g. Then consider h with $d_s(1, h) = p + 1$. Write h as the product of h' and s where $d_S(1, h') = p$ and $s \in S \cup S^{-1}$. This is possbile as we can choose h' as the product of the first p generators of h and then s is the $(p + 1)^{\text{st}}$ generator. Now, by induction Ah' is almost invariant. Hence, by the argument in the previous paragraph, the class of Ah is invariant in PG/FG. This completes the induction and we see that the equivalence class of A is invariant under every $g \in G$. So $A \in QG$ if and only if δc is finite

We have just shown that 0-cochains with finite coboundary correspond to elements of QG. So, taking the quotient QG/FG corresponds to taking the quotient of the 0-cochains with finite coboundary by finite 0-cochains. This is exactly what we noted $H^0_e(X)$ to be in Proposition 5.4. □

This equivalence is useful as the first definition of ends appears to be dependent on a choice of generating set. However, as the second definition does not use generating sets we can talk now refer to *the* number of ends of a group.

We now proceed with one more theory of ends, defined on any topological space. This generality is useful and the new definition also coincides with the previous definitions for simplicial complexes (hence Cayley graphs). This definition comes from Bridson and Haefliger [1, pp.144-145]

First we define some basic properties.

Definition 5.11. A map $f : X \to Y$ between topological spaces is said to be *proper* if $f^{-1}(C) \subset X$ is compact whenever $C \subset Y$ is compact.

Definition 5.12. Let X be a topological space. A *ray* in X is a continuous map $r : [0, \infty) \to X$. If $r_1, r_2 : [0, \infty) \to X$ are proper rays, then r_1 and r_2 are said to *converge to the same end* if for every compact $C \subset X$ there exists $N \in \mathbb{N}$ such that $r_1[N, \infty)$ and $r_2[N, \infty)$ are contained in the same path component of $X \setminus C$.

This defines an equivalence relation on continuous proper rays; the equivalence class of r is denoted by *end*(r) and the set of equivalence classes is denoted Ends(X). If the cardinality of Ends(X) is m, then X is said to have m ends.

Proposition 5.13. *For* Γ *a group and* S *a finite generating set for* Γ*:*
$e(\Gamma_S) = \sup\{n(K) : K \text{ a finite subgraph}\} = |\mathit{Ends}|(\Gamma)$

Proof: Suppose $\sup\{n(K)\} \geq N$ (Possibly $N =$ infinity)

Then there exists, K_N, with $n(K_N) \geq N$. That is, there exists a finite subgraph K_N of Γ such that $\Gamma \setminus K_N$ has at least N infinite components, C_i.

As the C_i are infinite there exists at least one ray in each. However, in $\Gamma \setminus K_n$ the C_i are not connected, and, in a Cayley graph (a metric space) this means they are not path connected and all of the rays are in separate equivalence classes. Hence $|\text{Ends}|(\Gamma) \geq N$.

Now suppose $|\text{Ends}|(\Gamma) \geq N$ then there exists a compact set K such that for any $M \in \mathbb{N}$, $r_i[M, \infty)$ and $r_j[M, \infty)$ lie in different path components (for $i \neq j$)

If there were fewer than N infinite components then at least two inequivalent rays r_i and r_j would lie in the same infinite path component. Hence there must be at least N infinite path components. □

Definition 5.14. Let X be a metric space. By definition, a *k-path* connecting x to y is a finite sequence of points $x = x_1, \ldots, x_n = y$ in X such that $d(x_i, x_{i+1}) \leq k$ for $i = 1, \ldots, n-1$.

Lemma 5.15. *Let X be a proper geodesic space and let $k > 0$. Let r_1 and r_2 be proper rays in X. Let $G_{x_0}(X)$ denote the set of geodesic rays issuing from $x_0 \in X$. Then:*

1. *end(r_1) = end(r_2) if and only if for every $R > 0$ there exists $T > 0$ such that $r_1(t)$ can be connected to $r_2(t)$ by taking a k-path in $X \setminus B(X_0, R)$ whenever $t > T$.*
2. *The natural map $G_{x_0}(X) \to$Ends(X) is surjective.*

Proof: Every compact subset of X is contained in an open ball about x_0 and vice versa, so one may replace compact sets by open balls $B(x_0, R)$ in the definitions of Ends(X).

Part (1) follows from this observation, because if $x_1, \ldots, x_n$ is a $k-$path connecting x_1 to x_n in $X \setminus B(x_0, R+k)$, then the concatenation of any choice of geodesics $[x_i, x_{i+1}]$ gives a continuous path from x_1 to x_n in $X \setminus B(x_0, R)$.

Now we show part (2). Let $r : [0, \infty) \to X$ be a proper ray. Let $c_n : [0, d_n] \to X$ be a geodesic joining x_0 to $r(n)$ where $d_n = d(x_0, r(n)$; extend c_n to be a constant on $[d_n, \infty)$. Because X is proper, the Arzela-Ascoli theorem [16, pp. 245] gives a subsequence of the c_n converging to a geodesic ray $c : [0, \infty) \to X$ and it is clear that *end*(c) = *end*(r). □

6 Properties of Ends

In this section we state and prove the central results of this paper. These are that

1. The number of ends of a space is a quasi-isometric invariant.
2. The number of ends of a group is invariant over taking finite index subgroups and taking quotients by finite subgroups.
3. The only possibilities for the number of ends of a group are 0,1,2 or infinity.
4. Finally we prove structure theorems for groups with 2 ends.

Definition 6.1. Let (X, d) be a metric space. A *geodesic ray* in X is a map $c : [0, \infty) \to X$ such that $d(c(t), c(t')) = |t - t'|$ for all $t, t' \geq 0$.

Lemma 6.2. *If X_1 and X_2 are proper geodesic spaces, every quasi-isometry $f : X_1 \to X_2$ induces a homeomorphism f_ϵ :Ends$(X_1) \to$Ends(X_2).*

Proof: Let r be a geodesic ray in X_1 and let $f_*(r)$ be a ray in X_2 obtained by concatenating some choice of geodesic segments $[fr(n), fr(n+1)], n \in \mathbb{N}$. Because f is a (λ, ϵ)-quasi-isometry, this is a proper geodesic ray. It is clear that $end(f_*(r))$ is independent of the choice of the geodesic segments $[fr(n), fr(n+1)]$.

Define f_ϵ : Ends$(X_1) \to$ Ends(X_2) by $end(r) \to end(f_*(r))$ for every geodesic ray r in X_1. The image under f of any k-path in X_1 is a $(\lambda k+\epsilon)$-path in X_2, so, by Lemma 5.15(1) we have that f_ϵ is well-defined on equivalence classes (and that it is continuous). Note that Lemma 5.15 (2) ensures that f_ϵ is defined on the whole of Ends(X_1).

Now, it is clear that if $f' : X_2 \to X_2$ are quasi-isometries then $f'_\epsilon f_\epsilon = (f'f)_\epsilon$, and if $f' : X_2 \to X_1$ is a quasi-isometric inverse for f, then $f'_\epsilon f_\epsilon = (f'f)_\epsilon$ is the identity map on Ends(X_1). □

The above result tells us that the number of ends of a space is invariant under quasi-isometry. Therefore using Lemma 3.18, we see that for the purposes of the study of ends, it is meaningful to talk about, Γ, the Cayley graph of a group. We can also now deduce that a group has the same number of ends as its Cayley graph because we saw that they are quasi-isometric.

Lemma 6.3. *The number of ends of the group $\mathbb{Z}$ is 2.*

Proof: The number of ends of G is invariant under homeomorphism and the Cayley graph for $(G, \{1\})$ is homeomorphic to $\mathbb{R}$.

Proposition 6.4. *If H is a subgroup of finite index in $G, e(G) = e(H)$.*

Proof: In the finitely generated case this follows from Lemma 3.19 and the fact that ends is a quasi-isometric invariant (Lemma 6.2).

For the general case we construct an isomorphism from QG/FG to QH/FH and hence show that they have the same number of ends.

So, consider A, an almost invariant subset of G. Suppose $h \in H$ then $x \in (A\cap H)+(A\cap H)h$ if and only if x is in exactly one of $A\cap H$ and $(A\cap H)h$, which is to say $x \in H$ and $x \in (A + Ah)$. Therefore, $(A \cap H) + (A \cap H)h = (A + Ah) \cap H$. Now, $(A + Ah)$ is finite for any $h \in H$, so $(A \cap H) + (A \cap H)h$ is also finite and $A \cap H$ is an almost invariant subset of H.

This gives a map, call it π, from QG to QH. Consider the induced map, $\phi : QG/FG \to QH/FH$.

$$\phi(A + FG) = ((A \cap H) + F(G \cap H)) = (A \cap H + FH).$$

The map ϕ is well defined as if A is almost equal to B, then A∩H is almost equal to B∩H. Also

$$\begin{aligned}\phi((A+B)+FG) &= ((A+B)\cap H)+FH \\ &= ((A\cap H)+(B\cap H))+FH \\ &= (A\cap H+FH)+(B\cap H+FH) \\ &= \phi(A+FG)+\phi(B+FG).\end{aligned}$$

So ϕ is a homomorphism. Choose a set of coset representatives (a left transversal) T for H in G.

For A in QG, if $A\cap H$ is finite then so is $Ag^{-1}\cap H$ (A is almost invariant so Ag^{-1} is almost equal to A), hence $A\cap Hg$ is. Letting g run through the finitely many elements of T we see that A (the union of all the $A\cap Hg$'s) is finite.

So, π maps only finite sets to finite sets. So the induced map ϕ maps only the identity (in QG/FG) to the identity (in QH/FH), that is, ϕ is injective.

Now, consider any almost invariant $B\subset H$. Define $A=BT$, so $A\cap H = B$. For any $g\in G$ and $t\in T$, as T is a left transversal we can write $tg=h_ts$ ($s\in T$).

However $A+Ag=\bigcup\limits_t(Bt+Btg)$ so by the above we see that this is $\bigcup\limits_t(Bt+Bh_ts)$. Noting that almost equality is an equivalence relation and as B is almost invariant $Bt+Bh_ts$ is finite. Also T is finite so we are taking a finite union so $A+Ag$ is finite. Thus, A is almost invariant and $\phi(A)=B$ so ϕ is surjective. □

Let A/K represent $\{aK : a\in A\}$

Lemma 6.5. *If K is a finite normal subgroup of G then: $e(G)=e(G/K)$)*

Proof: We will prove that

$$\frac{QG}{FG}\cong\frac{Q(G/K)}{F(G/K)}.$$

Consider the natural map $p:G\to G/K$ defined by $p(g)=gK$. This induces the maps,

$$p_t : PG\to P(G/K)\ \text{ by }\ p_t(A)=\{aK : a\in A\}$$

and

$$p_t^{-1} : P(G/K)\to PG\ \text{ by }\ p^{-1}(A/K)=\{a : aK\in A/K\}).$$

These give us the required isomorphism.

The maps are immediately seen to be homomorphisms. We want to show that they preserve Q and F and are two sided inverses.

Consider $B \subset G/K$. If gK is in B then a coset representative, use g, is in $p^{-1}(B)$. Thus gK is in $p_t p^{-1}(B)$. Similarly gK in $p_t p^{-1}(B)$ means gK in B. So $p_t p^{-1}(B) = B$. Also, a similar argument for $A \subset G$ gives $p^{-1}p_t(A) = AK$.

If B is almost invariant then consider $p^{-1}(B)$. For each element of B we have at most $|K|$ elements in $p^{-1}(B)$. Similarly for $p^{-1}(B) + p^{-1}(B)g$ for $g \in G$. So, as K finite and B are almost invariant, $p^{-1}(B) + p^{-1}(B)g$ will be finite for all $g \in G$. Hence $p^{-1}(B)$ is almost invariant.

Now suppose A is almost invariant. Then $A + Ak$ is finite for all $k \in K$. Also, there are only finitely many such k so A is almost equal to AK. This means that AK is almost invariant. Now $g \in AK$ if and only if $gK \in A/K$ and similarly for AKg and $(A/K)gK$. So $(A/K) + (A/K)gK$ finite (as K finite) and $p_t(AK) = p_t(A)$ is almost invariant.

We have maps preserving F and Q, so the induced maps between QG/FG and $Q(G/K)/F(G/K)$ are, as before, homomorphisms and are two sided inverses, and hence isomorphisms. □

Lemma 6.6. *Let $A_0, A_1 \in QG$. For almost all $g \in A_0$, either $gA_1 \subset A_0$ or $gA_1^* \subset A_0$*

Proof: Choose a finite set, S, of generators of G, and consider (as in Proposition 5.10) the action of G on Γ_S. Fix connected finite subgraphs C_i of Γ_S containing δA_i.

Now, A_0 is almost invariant. Thus given $c \in G$, for almost all $g \in A_0$ we have $gc \in A_0$. If not then running through the infinitely many g with gc not in A would yield infinitely many elements of $A_0 + A_0 c$.

Given a vertex c of C_1, for almost all $gc \in A_0$, $g \in A_0$. As C_i is finite, $gC_i \cap C_i = \emptyset$ for almost all $g \in G$. Hence for almost all $g \in A_0$, we have $gC_1 \cap C_0 = \emptyset$ and $gc \in A_0$ for each vertex c of C_1. Thus, by the finiteness of C_1, we can choose some $g \in G$ such that these properties hold for all c in C_1.

Now, for any collection, A, of vertices of Γ, let $\overline{A}$ denote the maximal subgraph of Γ with vertex set equal to A. Each component E of $\overline{A_1}$ or $\overline{A_1^*}$ contains a vertex of C_1, so gE meets A_0 (by construction of g). If gE also meets A_0^* (as C_0 contains all edges in the boundary of A_0, any set which has points in A_0 and A_0^* meets C_0), it meets C_0. However C_0 is connected and disjoint from gC_1, so lies in a single component gE. Thus A_0^* cannot meet both gA_1 and gA_1^*. This completes the proof. □

Definition 6.7. For a group G, and $A \subset G$, the isotropy group H is defined to be $H := \{h \in G : hA \stackrel{a}{=} A\}$.

For a subset A of G let A^* denote the complement of A in G.

Proposition 6.8. *Suppose G is finitely generated, and $A \in QG$ is such that both A and A^* are infinite, and that the isotropy subgroup of A is infinite. Then G has an infinite cyclic subgroup of finite index.*

Proof: Elements of H are either in A or A^* so if $A \cap H$ is finite then, as H infinite, $A^* \cap H$ is infinite. So, by interchanging A and A^* if necessary, we may assume $A \cap H$ is infinite as, by Lemma 5.7 if A is almost invariant then A^* is almost invariant. We may also adjoin 1 to A.

By Lemma 6.6, for almost all $g \in A$ either $gA \subset A \setminus \{1\}$ or $gA^* \subset A \setminus \{1\}$. We want some $c \in H \cap A$ satisfying one of these. However, there are only finitely many elements of A that do not, so as $H \cap A$ is infinite it must contain such a c. Now, $c \in H$ gives $cA \overset{a}{=} A$, and, A^* is almost invariant so $cA^* \cap A \setminus \{1\}$ if finite. This means that $cA \subset A \setminus \{1\}$. We now show that c generates the required subgroup.

Now, $cA \subset A \setminus \{1\}$ so $c^2A \subset cA \setminus \{c\} \subset cA$ and if $n > 0$ then

$$c^nA \subset c^{n-1}A \subset \ldots \subset cA \subset A \setminus \{1\}.$$

Thus, 1 is not in c^nA and hence $c^n \neq 1$ for all $n > 0$, so c has infinite order.

Recall that $1 \in A$ so $c^n \in A$ for all $n > 0$. Also $c^nA \subset A \setminus \{1\}$ and $c^{-n} \in A$ would imply that $1 \in c^nA$. We deduce that $c^{-n} \in A^*$ for all $n > 0$.

Suppose, for a contradiction $d \in \bigcap\{c^nA : n > 0\}$. Then $c^{-n} \in Ad^{-1}$ for all $n > 0$, $c^{-n} \in Ad^{-1} + A$, since $c^{-n} \notin A$. However $Ad^{-1} + A$ is finite and all the c^{-n} are distinct so this is a contradiction. Hence, $\bigcap\{c^nA : n > 0\} = \emptyset$.

Therefore $A = (A \setminus cA) \cup (cA \setminus c^2A) \cup \ldots$ or, more formally,

$$A = \bigcup\{c^nA \setminus c^{n+1}A : n > 0\}$$

Which can be written as $\bigcup\{c^n(A \setminus cA) : n > 0\}$.

Now, $A \setminus cA$ is finite (as $c \in H$ so $A \overset{a}{=} cA$). Also $\bigcup\{c^n(A \setminus cA) : n > 0\} \subset \bigcup\{\langle c\rangle a : a \in A \setminus cA\}$. This shows that A is contained in the union of finitely many cosets of $\langle c\rangle$ in G.

Replacing c by c^{-1} we see that A^* is the union of finitely many cosets also (by repeating the process we used for A). So, as $G = A \cup A^*$ we see that $\langle c\rangle$ has finite index in G. □

Proposition 6.9. *If G is finitely generated, $e(G) = 0, 1, 2$ or ∞.*

Proof: Suppose $e(G) \neq 0, 1$ or ∞. Then, as $e(G) \neq 0$, the group G is infinite. By definition of $e(G) \neq \infty$), the group QG/FG is finite . Also,

$e(G) \neq 1$ so QG/FG contains at least 4 elements. One of these is the equivalence class of $\emptyset$, another element is the equivalence class of G. Picking one of the remaining elements we get an A such that both A and A^* are infinite.

Suppose, for a contradiction, H has infinite index in G. Then there would be an infinite set of cosets of H, $(g_1H, g_2H, \ldots)$. Taking a coset representative for each we get g_iA not almost equal to g_jA for all $i \neq j$. These will all be almost invariant which would give infinitely many elements of QG/FG which contradicts our hypothesis. Thus the isotropy group, H, has finite index in G.

Now we have satisfied the conditions of Proposition 6.8 so G has an infinite cyclic subgroup, H, of finite index. Also, H is isomorphic to $\mathbb{Z}$. We noted in Lemma 6.3 that $e(\mathbb{Z}) = 2$ so $e(H) = 2$. However, Proposition 6.4 gives $e(G) = e(H) = 2$. This completes the proof. □

Given the above result, it is natural to ask if we can find structure theorems for groups with a specific number of ends. We have already noted that groups with 0 ends are finite.

Theorem 6.10. *Let G be a finitely generated group. The following are equivalent:*

1. $e(G) = 2$,
2. *G has an infinite cyclic subgroup of finite index,*
3. *G has a finite normal subgroup with quotient* $\mathbb{Z}$ *or* $\mathbb{Z}_2 * \mathbb{Z}_2$,
4. $G = F*_F$ *with F finite, or* $G = A *_F B$ *with F finite and*

$$|A : F| = |B : F| = 2.$$

Proof:

(1) ⇒ (2): Note that the proof of Proposition 6.8 tells us that,

(a) the isotropy subgroup, H, is infinite and that

(b) we can find A and A^* in QG/FG both infinite. That is, we can satisfy the conditions for Proposition 6.9 and so G has an infinite cyclic group of finite index.

(2) ⇒ (1): Lemma 6.3 gives $e(\mathbb{Z}) = 2$, and Proposition 6.4 gives

$$e(G) = e(\mathbb{Z}) = 2.$$

(3) ⇒ (4):

(a) If F is a finite normal subgroup of G with quotient $\mathbb{Z}$, then (by [7]) there exists $K \cong \mathbb{Z}$ such that G is a semi-direct product of F by K. If the presentation of F is $\langle Y \,|\, T\rangle$ then the presentation of this is $\langle Y, t | \, T, t^{-1}at = f(a)$, for a in $F\rangle$ that is $G = F{*}_F$.

(b) Suppose G/F is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. Then take the pull-back of the generators of the $\mathbb{Z}_2$ factors. This gives us elements whose squares are in F. Since F is normal we get two subgroups A, B of G with F as an index 2 subgroup. Also the intersection of A and B is F.

Construct a homomorphism, $\psi : A *_F B \to G$ by $\psi(A) = A$ and $\psi(B) = B$. By the substitution test ([7]) and the fact that F is normal in G this extends to a (surjective) homomorphism. Also suppose $\psi(\prod_{i,j} a_i b_j) = 1$. Then as ψ is the identity on A and B we get that $(\prod_{i,j} a_i b_j) = 1$ and so ψ is injective. Hence ψ is an isomorphism. We also noted that $|A : F| = |B : F| = 2$ and this proves that (3) ⇒ (4).

(4) ⇒ (3): If $G = F{*}_F$ with F finite, then F is normal in G with quotient $\mathbb{Z}$.

If $G = A *_F B$, with F finite and $|A : F| = |B : F| = 2$, then F is normal in A and in B. Hence F is normal in G and

$$G/F \cong (A/F) * (B/F) \cong \mathbb{Z}_2 * \mathbb{Z}_2.$$

(3) ⇒ (1): By [7] $\mathbb{Z}_2 * \mathbb{Z}_2 \cong D_\infty$. So, by Lemma 6.5 $e(G) = e(D_\infty) = 2$.

(2) ⇒ (3): We use the existence of an infinite cyclic subgroup, C, of finite index in G to construct an infinite cyclic group K of finite index in G which is also normal in G. Let $K := \bigcap_{g \in G} g^{-1}Cg$. Then

$$h^{-1}Kh = h^{-1}\big(\bigcap_{g\in G} g^{-1}Cg\big)h = \big(\bigcap_{g\in G} h^{-1}g^{-1}Cgh\big) = \big(\bigcap_{g\in G} g^{-1}Cg\big) = K,$$

so K is normal in G. Also, as C has finite index this is a finite intersection. Hence K is infinite cyclic and has finite index in G.

Let G act on K by conjugation and define H to be the centralizer of K in G. However, for each g in G, conjugating by g defines an automorphism of K (as K is normal in G). Thus, as $K \cong \mathbb{Z}$, there are only 2 distinct such automorphisms, either preserving the generator or sending it to its inverse. So $|G : H| \leq 2$ hence $H \lhd G$ and H infinite.

Now, H is finitely generated as H is a finite index subgroup of a finitely generated group. Note that $K \leq Z(H)$ so as K has finite index in G $Z(H)$ has finite index in H. However, $Z(H)$ has finite index implies that its commutator subgroup, H', is finite (see, for example, [19]).

Now, as G has a cyclic group of finite index H/H' has rank 1. So, there exists an onto homomorphism $\phi : H \to \mathbb{Z}$ with finite kernel L. If $G = H$ then $G/L \cong \mathbb{Z}$ and the result is proved.

Suppose $G \neq H$. Note that L is the torsion subset of H as H has rank 1. L is a normal subgroup of H as it is the kernel of a homomorphism. Now, as automorphisms preserve the order of elements, L is characteristic in H. Note that the action of G on H by conjugation gives rise to an automorphism on H. As L is characteristic in H it is invariant under this action so L is normal in G.

By the Third Isomorphism Theorem and the fact that $|G : H| = 2$ we have

$$\frac{G/L}{H/L} \cong \mathbb{Z}_2$$

Since $G \neq H$ there exists an element, $g \in G$, which does not commute with some element $k \in K$. Now, k must be non trivial in G/L because it has infinite order. However, K is normal in G so $g^{-1}kg \in K$. As g doesn't commute with k the commutator $k^{-1}g^{-1}kg$ is not the identity in K, so has infinite order. Therefore it is a non trivial commutator in G/L; hence G/L is non-abelian.

Let $x \in \frac{G/L}{H/L}$ be non trivial and let $H/L = \langle y \rangle$ (H/L is cyclic). Then G/L is generated by x and y. G/L is non abelian so $x^{-1}yx \neq y$. However as y has infinite order we must have

$$x^{-1}yx = y^{-1}. \tag{3}$$

Now, $x^2 \in H/L$ so

$$x^2 = y^k, \tag{4}$$

for some k. However (3) gives

$$x^{-1}y^k x = y^{-k} \overset{(4)}{=} x^{-2}.$$

That is

$$x^{-1}y^k x = x^{-2}$$

hence

$$x^2 = y^{-k}.$$

This forces $k = 0$ hence $x^2 = 1$ in G/L.

Now, $G/L = \langle x\rangle\langle y\rangle$ and $\langle x\rangle \cap \langle y\rangle = \{1\}$ but also $\langle y\rangle \lhd G/L$. By [7],

$$G/L \cong \mathbb{Z} \rtimes_\phi \langle x\rangle$$

with $\phi(y) = x^{-1}yx = y^{-1}$. This has presentation

$$\langle x, y|\ x^2, x^{-1}yxy\rangle = D_\infty.$$

By [7] D_∞ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

This completes the proof of Theorem 6.10.

7 Applications and More Recent Results

Many interesting results have been proved in the area of infinite (discrete) group theory and this paper gives only a brief discussion of a small part of that theory.

With so few possibilities for $e(\Gamma)$ we naturally ask for characterisations of the different numbers of ends. Having 0 ends is equivalent to the group being finite. If $e(X) \geq 1$ then it is infinite. Also, we have Theorem 6.10 which characterises groups with 2 ends.

A natural question is to ask what can be said about groups with 1 or infinitely many ends.

Although the proof lies beyond the scope of this work, the following theorem, due to Stallings [22], is fundamental to the field.

Theorem 7.1 (Stallings). *Let G be a finitely generated group with infinitely many ends.*

1. *If G is torsion-free then G is a non-trivial free product, otherwise*

2. *G is a non-trivial free product with amalgamation, with finite amalgamated subgroup.*

There is no such classification for groups with only 1 end.

However, these results allow us to break up some of the theory into smaller parts. We might, for example, consider some group of isometries of a space X and then, under the right conditions, the Švarc-Milnor Lemma 3.16 would tell us the number of ends of G.

If G has 2 ends we then apply Theorem 6.10 and we already know some of the structure of G. If G has infinitely many ends then Stallings' Theorem tells us that G is a free product with amalgamation with finite amalgamated subgroup.

An obvious question posed by the above is, given a group, Γ with $e(\Gamma) > 1$, can we use the above theorems to write Γ as a finite product of groups with 0 or 1 end?

In the case of Γ having 2 ends, as Γ has a subgroup isomorphic to $\mathbb{Z}$ of finite index, we see that writing $\Gamma = A *_C B$ or $\Gamma = F *_F$ gives a free product of finite groups.

If Γ has infinitely many ends then the answer is more subtle. It is possible that $\Gamma = A *_C B$ where one, or even both of, A and B have infinitely many (or 2) ends.

An interesting question then is, can this process repeat for ever? Is it possible that Γ will split infinitely many times, each product being of groups, at least one of which also have infinitely many ends?

This question can be posed more succinctly by considering the question of *accessibility.* We can think of splitting a group over a finite subgroup, that is writing it as an amalgamated free product, as a sort of factorisation.

Now, we know that if $e(\Gamma) \geq 2$ then there exists such a factorisation. The natural question is whether such a process can continue forever or whether it must stop.

Of course, with groups which are not finitely generated the answer is that the process can continue forever. Consider the free group on $\aleph_0$ generators. This has infinitely many ends and it is clear that the process of splitting into free products can continue indefinitely.

This can be formalised with following definitions which come from Scott and Wall ([18, pp. 189]).

Definition 7.2. A finitely generated group Γ with at most 1 end is *0-accessible.*

If Γ has more than 1 end we can use Stallings' Theorem or Theorem 6.10 to write Γ as a free product with amalgamation with finite amalgamated subgroup. If both factors then have at most one end we say Γ is 1-accessible.

We then make the following recursive

Definition 7.3. Define Γ to be $n-$*accessible* if Γ splits over a finite subgroup with each of the factor groups $(n-1)$-accessible. A group is said to be *accessible* if it is $n-$accessible for some n.

Theorem 7.4. *Gruško's Theorem [8, pp. 365-372]: Let F be a finitely generated free group, $G = G_1 * G_2$ and let $\phi : F \rightarrow G$ be an epimorphism.*

Then there are subgroups F_1 and F_2 of F such that $F = F_1 * F_2$ and $\phi(F_i) = G_i$.

This tells us that if G can be generated by n elements, then there exists a set of n generators for G with each element in G_1 or G_2.

In particular if G is finitely generated and torsion-free then G is accessible, by induction on the number of generators (the case $n = 1$ being $G \cong \mathbb{Z}$ so we can apply Theorem 6.10).

We can use these ideas to get a surprising result:

Corollory 7.5. *[21] Suppose G is a finitely generated torsion-free group and suppose G has a free subgroup, F, of finite index. Then G is free.*

Proof: Gruško's Theorem tells us that each factor group has strictly fewer generators than G. So, using this fact and Stalling's theorem we see that G is accessible.

We now proceed by induction on n, where G has n generators. Suppose G has 1 generator then G is cyclic and hence free.

If G has more than one generator then it has a free subgroup of finite index. If this is $\mathbb{Z}$ then by Proposition 6.4 G has 2 ends. Otherwise $F \leq G$ where F is the free group on m generators so by Remark 5.3 and Proposition 6.4 G has infinitely many ends.

If $e(G) = 2$ then, by 6.10 (part (4)) $G = \{1\} * \{1\}$ as G torsion free.

If $e(G) = \infty$ then $G = A * B$, a non trivial free product, by Stallings'. By Grushko's theorem, A and B are generated by strictly fewer elements. Subgroups of free groups are free, hence the intersection of A with the free subgroup is a free subgroup. So by induction A is free. Similarly for B. So $G = A * B$ is free. This completes the proof.

In [2, pp. 449-457], Dunwoody proved that finitely presented groups are accessible, whilst in [3, pp. 75-78] he provided a counterexample for finitely generated groups.

We've seen with Stallings' Theorem that groups which split over finite groups give rise to groups with more than one end. In work around the late 1970s, Bass and Serre [17] developed a theory of more general splittings with regards to groups acting on simplicial trees. This gave rise to the notion of 'graphs of groups', which are generalisations of free products with amalgamation and HNN extensions (although formally, they reduce to repeated applications of these two concepts). Given this framework, and in the context of groups acting on R-trees (a generalisation of simplicial trees), and also in the context of the canonical JSJ decomposition of 3-manifolds, Rips and

Sela [15, pp. 53-109] developed a theory of canonical splittings of finitely presented groups over virtually cyclic groups (the so-called JSJ decomposition of a finitely presented group).

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